RESOLVING SINGULARITIES VIA LOCAL QUADRATIC TRANSFORMATIONS*

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Let V be an analytic subset of a domain $U' \subset \mathbf{C}^m$ of pure dimension n, and let $p \in \operatorname{Sg} V$, the singular locus of V, be such that near p, $\operatorname{Sg} V$ is an $n-1$ manifold. We may select coordinates (x_i), defined in a neighborhood $U \subset U'$ of p in $\mathbf{C}^m$, such that p becomes the origin and $\operatorname{Sg} V \cap U = \{x \in U : x_i = 0 \text{ for } i \geqq n\}$. For $q \in \operatorname{Sg} V \cap U$ set $V_q = \{x \in V \cap U : x_i = q_i$ for $i < n\}$. The (x_i) may be chosen so that V_q has dimension one. If this is the case we say that (x_i) is adapted to V at p. Each of the V_q can be resolved by a sequence of local quadratic transformations; however, the series may vary greatly with q. The idea of resolution via local quadratic transformations can be extended from the one-dimensional case to varieties such as V. (See Definitions 1 and 2 below.) In [*1*] Zariski showed that if V is a hypersurface then the existence of a resolution for V via local quadratic transformations is one of a number of equivalent criteria for V to be equisingular along $\operatorname{Sg} V$ at p. Such a resolution will induce resolutions of the V_q and, in the hypersurface case, these resolutions are essentially the same. In [*2*] a generalization of Zariski's results to varieties of higher embedding dimension was begun, and three types of equisingularity were defined: weak, strong, and residual. Each type generalized a part of Zariski's results, residual equisingularity generalizing those parts of Zariski's results which were concerned with the resolution of V. The purpose of this note is to extend the results of [*2*] on residual equisingularity. We have three main results. First we show that if V is residually equisingular at p then V is weakly equisingular at p. This allows us to obtain V as the image of a particularly simple type of holomorphic map from a polydisk, and so to show that near a weakly equisingular point $p \in \operatorname{Sg} V$, $\{q \in \operatorname{Sg} V : V$ is not residually equisingular at $q\}$ is contained in proper analytic subset $A \subset \operatorname{Sg} V$. In [*1*] the similarity of the resolutions of the V_q was expressed by the property of tangential stability (see Definition 4 below). We show below that

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if V is residually equisingular at p then there is a resolution of V which induces tangentially stable resolutions for (V_q).

Finally we explore the relations between the various types of equisingularity. In [2] we show that strong implies weak. Residual also implies weak as noted above. Our third result is a pair of examples which show that even in the irreducible case strong and residual equisingularity are completely independent notions; in general neither implies the other.

In [4] Zariski describes his study of equisingularity for hypersurfaces as a first step in a search for a general equisingular stratification. Our results indicate that, in general, there are a number of different choices for the strata in codimension 1.

§1. Let V, U and $p \in \operatorname{Sg} V$ be as above, and let (x_i) be coordinates adapted to V near p. For such coordinates we define three types of maps. Let W be an open subset of U.

i) $R: W \to U$ by $R(x) = (x_1, \cdots, x_n, x_{n+1} - r_1(x_1, \cdots, x_n), \cdots, x_m - - r_{m-n}(x_1, \cdots, x_n))$.

ii) $S: W \to U$ by $S(x) = (x_1, \cdots, x_{n-1}, s_0(x), \cdots, s_{m-n}(x))$

$$s_i(x) = \sum_{j=n}^{m} a_{ij} x_j,\ a_{ij} \in \mathbf{C} \text{ and } \operatorname{rk}(a_{ij}) = m - n\,.$$

iii) $T: W \to U$ by $T(x) = (x_1, \cdots, x_n, x_{n+1}, x_n, \cdots, x_m \cdot x_n)$.

Definition 1. A *connected sequence of local quadratic transformatons* is a set of maps $F_j: W_j \to W_{j-1}, j = 1, \cdots, l$, such that

a) $F = F_1 \circ \cdots \circ F_l$ is well defined.

b) On each component $W_{j,k}$ of W_j, F_j is of type i), ii), or iii).

c) If $F_j(W_{j,k_1}) \cap F_j(W_{j,k_2})$ is not empty then $F_j \big| W_{j,k_1}$ and $F_j \big| W_{j,k_2}$ are of the same type.

Suppose that a connected sequence of quadratic transformations is given and $V \subset$ image F. Set $F_J = F_1 \circ \cdots \circ F_j$ for $1 \leqq j \leqq l$ and set $V_j = F_J^{-1}(V)$. (Of course if a type iii map occurs we take only those components of the inverse image which lie outside $\{x \in \mathbf{C}^m : x_n = 0\}$.)

Definition 2. V is *residually equisingular at* $p \in \operatorname{Sg} V$ if there is a connected sequence of quadratic transformations such that

a) $F \big| V_L$ is a resolution of p in a neighborhood of p.

b) $F^{-1}(\operatorname{Sg} V)$ is a manifold and F restricted to each component of $F^{-1}(\operatorname{Sg} V)$ is biholomorphic.

c) For all j each component of V_j is either a manifold or is singular at every point of $F_J^{-1}(\operatorname{Sg} V)$ which it contains.

Let (V_k) be a set of representatives for the irreducible branches of V at p and let $C_i(V,p)$, $i = 3,4,5$, be the Whitney tangent cones to V at p. (For a definition see [3].)

Definition 3. V is *weakly equisingular at* $p \in \operatorname{Sg} V$ if

a) $\dim C_4(V,p) = n$.

b) for all $k_1 \neq k_2$, $V_{k_1} \cap V_{k_2} = \operatorname{Sg} V$ near p.

Proposition 1. If V is residually equisingular at $p \in \operatorname{Sg} V$ then V is weakly equisingular at p.

Proof. First consider the case when $l = 1$ and V is irreducible at p. In this case $F = F_1$ can be assumed to be of type iii. $F \mid V_1$ has a weakly holomorphic inverse which extends to p so $F^{-1}(p)$ consists of a single point, q. Let $(e_i) \subset T(\mathbf{C}^m, q)$ be the basis associated with (x_i). From 2b it is immediate that for each i, $1 \leqq i \leqq n-1$, we have $e_i \in T(V_1, q)$. Observe that if $v = (v_1, \cdots, v_m) \in T(V_1, q)$ then $v_n = 0$, for if not $F_*(q)$ would have rank n and V would be nonsingular at p. Thus there is $v \in T(V_1, p)$ with $v_i \neq 0$ for some $i > n$. Suppose $v_{n+1} \neq 0$. The projection $x \to (x_1, \cdots, x_{n-1}, x_{n+1})$ restricted to a neighborhood of q in V_1 will be biholomorphic. We write its inverse as follows, $G(y) = (y_1, \cdots, y_{n-1}, g_1(y), \cdots, g_{m-n+1}(y))$. Set $G^{-1}(q) = r$. Notice that the projection $y \to (y_1, \cdots, y_{n-1})$ restricted to a neighborhood of r in $(F \circ G)^{-1}(\operatorname{Sg} V)$ maps that neighborhood biholomorphically onto a neighborhood of 0 in $\mathbf{C}^{n-1}$. Thus, composing G with a change of coordinates H of the form $H(y) = (y_1, \cdots, y_{n-1}, y_n + h(y_1, \cdots, y_{n-1}))$, we may assume $(F \circ G)^{-1}(\operatorname{Sg} V) \subset \{y \in \mathbf{C}^n : y_n = 0\}$. This change of coordinates does not affect the form of G as given above, and we have $g_1 \mid \{y \in \mathbf{C}^n : y_n = 0\} \equiv 0$. At any point r' of the domain of G the jacobian of $F \circ G$ has the following block form;

$$J(F \circ G)(r') = \begin{bmatrix} I_{n-1} & 0 \\ \dfrac{\partial g_1}{\partial y_i}(r') & \dfrac{\partial g_1}{\partial y_n}(r') \\ \dfrac{\partial}{\partial y_i}(g_1 g_j)(r') & \dfrac{\partial}{\partial y_n}(g_1 g_j)(r') \end{bmatrix}.$$

Here I_{n-1} is the $(n-1) \times (n-1)$ identity matrix. Let (δ_i) be the basis for $T(\mathbf{C}^n, r)$ associated with (y_i). $C_4(V,p)$ is the fiber over p of the closure of the tangent bundle $T(V - \operatorname{Sg} V)$ in $T(\mathbf{C}^m)$. Let $p' \in V - \operatorname{Sg} V$ and $r' = (F \circ G)^{-1}(p')$. Since F is a resolution of V, $F \circ G$ is biholomorphic near r' and the vectors $u_i(p') = (F \circ G)_*(r')(\delta_i)\operatorname{span}(T(V,p')$. Notice that for $i < n$, $u_i(p) = \lim_{p' \to p} u_i(p') = e_i + \sum_{j \geq 2} g_j(r)(\partial/\partial y_i g_1)(r)e_{n+j-1}$. Let $u(p') = \sum a_i(p')u_i(p')$ be a sequence of vectors such that $u(p) = \lim_{p' \to p} u(p')$ determines an element of $C_4(V, p)$. Since $a_n(p')u_n(p')$ has its first $n-1$ coordinates equal to zero, we see from the above that for $i < n$, $a_i(p) = \lim_{p' \to p} a_i(p')$ must exist and $u(p) = \sum_{i<n} a_i(p)u_i(p) + \lim_{p' \to p} a_n(p')u_n(p')$ so the final limit must also exist.

$$u_n(p') = \frac{\partial g_1}{\partial y_n}(r')\left[e_n + \sum_{j \geq 2} g_j(r')e_{n+j-1}\right] + g_1(r') \sum_{j \geq 2} \frac{\partial g_j}{\partial y_n}(r')e_{n+j-1}$$

so we have

$$\tilde{a}_n(p) = \lim_{\substack{p' \to p \\ r' \to r}} a_n(p')\frac{\partial g_1}{\partial y_n}(r')$$

finite, since $g_1 \mid \{y \in \mathbf{C}^n : y_n = 0\} \equiv 0$. It follows immediately that $a_n(p)u_n(p) = \tilde{a}_n(p)\left[e_n + \sum_{j \geq 2} g_j(r)e_{n+j-1}\right]$. Thus we have exhibited $C_4(V, p)$ explicitly as an n-plane. Since V has only one component, 3b is vacuous and this case is complete.

Next we consider the case when V is irreducible at p but $l > 1$. As above set $q_j = F_J^{-1}(p)$. We may assume that F_l is of type iii, that $q_l \in \{x \in \mathbf{C}^m : x_n = 0\}$, and that V_{l-1} is singular at q_{l-1}. $F_{L-1}^{-1}(\operatorname{Sg} V)$ is an $n-1$ manifold near q_{l-1} and V_{l-1} must be singular at every point of it. Since F_l is of type iii this means that $F_{L-1}^{-1}(\operatorname{Sg} V) = \{x \in \mathbf{C}^m : x_i = 0 \text{ for } i \geq n) = \operatorname{Sg} V_{l-1}$ near q_{l-1}. $F_l : V_l \to V_{l-1}$ is a resolution of the type considered above so we may assume that the projection $x \to (x_1, \cdots, x_n)$ is transverse to $C_4(V_{l-1}, q_{l-1})$. V_{q-1} is clearly irreducible at q_{l-1} so from [3] we know that this projection can be used to construct a Puiseux series normalization for V_{l-1} at q_{l-1}, that is, if we replace V_{l-1} by a suitable neighborhood of q_{l-1} then we can find a polydisk $D \subset \mathbf{C}^n$ centered at 0, and a holomorphic homeomorphism $f : D \to V_{l-1}$ such that

$$f(y) = (y_1, \cdots, y_{n-1}, y_n^v, f_1(y), \cdots, f_{m-n}(y))$$

$$f_i(y) = \sum_{j \geq v} f_{i,j}(y_1, \cdots, y_{n-1})y_n^j.$$

Now we continue working back toward V. Suppose F_j, $j_0 < j < l$, are all of type i or ii but F_{j_0} is of type iii. Set $g = F_{j_0+1} \circ \cdots \circ F_{l-1}$, then we have

$$g(y) = (y_1, \cdots, y_{n-1}, g_0(y), \cdots, g_{m-n}(y)).$$

We claim that $g_0(y) = y_n^\alpha \cdot \tilde{g}_0(y)$ near 0, where α is an integer greater than 2 and $\tilde{g}(0) \neq 0$. Suppose not. The set $X = \{y \in D : g_0(y) = 0\}$ is an analytic set of pure dimension $n-1$. Let Y denote a component of X

containing 0 other than $D' = \{y \in D: y_n = 0\}$. Since $F: V_l \to V$ is a resolution we must have $F: V_l - F^{-1}(\mathrm{Sg}\, V) \to V - \mathrm{Sg}\, V$ biholomorphic. From this it is easy to see, by downward induction, that for each j, $F_j: V_j - F_J^{-1}(\mathrm{Sg}\, V) \to V_{j-1} - F_{j-1}^{-1}(\mathrm{Sg}\, V)$ is biholomorphic so $F_{j_0} \circ g: D - D' \to V_{j_0-1} - F_{J_0-1}(\mathrm{Sg}\, V)$ is also biholomorphic. Now it is clear that $Y \cap \{y \in D: y_i = 0 \text{ for } i < n\} = (0)$ so the map $\pi: Y \to D$ defined by $y \to (y_1, \cdots, y_{n-1})$ gives a branched covering. Again using the fact that $F_{j_0} \circ g \mid D - D'$ is biholomorphic we see that π must be of degree 1 so, by the nullstellensatz, g_0 must be divisible by a function of the form $(y_n - \alpha(y_1, \cdots, y_{n-1}))^k$. Observe that there can only be one such component Y, for if there were two then $F_{j_0} \circ g \mid D - D'$ would not be one-to-one. Next suppose that $D' \subset X$, then $Y \cup D' \subset (F_{j_0} \circ g)^{-1}(F_{J_0}(\mathrm{Sg}\ V) = g^{-1}(\mathrm{Sg}\, V_{l-1}) = D'$ so we again come to a contradiction. Finally if Y is the only component of X then since $\mathrm{rk}(F_{j_0} \circ g)_*(0) < n$ we must have $k > 1$, but this would give $\mathrm{rk}(F_{j_0} \circ g)_* < n$ at every point of Y. Thus we see that the only possibility is that $X = D'$ and so, again by the nullstellensatz, g_0 must have the desired form. We may now calculate $C_4(V_{j_0-1}, q_{j_0-1})$ directly as above, and show it is an n-plane. By repeating this argument as necessary we can eventually show that $C_4(V, p)$ is an n-plane. Since 3b is still vacuous we are finished with this case.

Finally let us consider the general case when V is reducible at p. First observe that each component V_k of V will have a resolution as above, so since $C_4(V, p) = U_k C_4(V_k, p)$ we need only check that the branches intersect properly. Each V_k gives rise to a separate connected component of V, and from 1c we see that the only way the images of two components can come to intersect is if both are subjected to a type iii map. In this case the components will intersect in a neighborhood of the origin in $\{x \in \mathbf{C}^m: x_i = 0 \text{ for } i \geqq n\}$. Using this observation at each stage of the resolution it is easy to see that 3b must hold, so the proof is complete.

Let $p \in \mathrm{Sg}\, V$ be such that V is weakly equisingular at p. We wish to describe the set of points $q \in \mathrm{Sg}\, V$ near p such that V is not residually equisingular at q. As above we know that we can select coordinates (x_i) adapted to V at p and representatives V_k for the irreducible components of the germ of V at p, defined in some neighborhood U of p, such that each component has a Puiseux series normalization $f_k: D \to V_k$. (The construction in [2] shows that all f_k may be assumed to have the same domain.) To fix notation let us write $f_k(y) = (y_1, \cdots, y_{n-1}, y_n^{v_k}, f_{1,k}(y), \cdots, f_{m-n,k}(y))$. Let K be the field of meromorphic functions on D'. We may view $y_n^{V_k}, f_{1,k}, \cdots$ as elements of $K[[y_n]]$. If $(g_0, \cdots, g_{m-n})$ are elements of $K[[y_n]]$, the inverse of a map of type i or ii corresponds to an arithmetic operation on $(g_0, \cdots, g_{m-n})$

giving rise to a new $m-n+1$-tuple. The inverse of a type iii map will do the same thing if $\operatorname{ord} g_0 = \min \operatorname{ord} g_i$. If F is a map whose inverse defines an operation on (g_i) we write $F^*: (g_i) \to (g_i)$. If $(g_{1,j}),(g_{2,i})$ are two $m-n+1$-tuples of elements of $K[[y_n]]$, we may write $g_{1,i} = \sum_l g_{1,i}^l y_n^l$ and similarly for $g_{2,i}$. We say that $(g_{1,i})$ and $(g_{2,i})$ intersect formally at p if for all i $g_{1,i}^0 = g_{2,i}^0$.

Proposition 2. $\{q \in \operatorname{Sg} V \cap U: V$ is not residually equisingular at $q\}$ is contained in a proper analytic subset $A \subset \operatorname{Sg} V \cup U$.

Proof. We begin by constructing a sequence of operations on the last $m-n+1$ terms of each f_k which is analogous to a resolution of V. We will write $({}_lf_{i,k})$ $i = 0,\cdots,m-n$ for the $m-n+1$-tuple obtained from $(y_n^{v_k},f_{1,k},\cdots,f_{m-n,k})$ after the first l operations. We begin by dividing repeatedly by y_n until $\operatorname{ord}{}_lf_{i,k} < v_k$ for some i,k. Now there are two possibilities. If $\operatorname{ord}{}_lf_{i,k} = 0$, we partition the set of $({}_lf_{i,k})$, grouping together those that intersect formally at p. Using a type i map on each group we may assume that $\min \operatorname{ord}{}_{l+1}f_{i,k} > 0$. Next using a type ii map on each group we may assume in addition that $\operatorname{ord}{}_{l+2}f_{0,k} \leqq \min_i {}_{l+1}f_{i,k}$ for all k. Now we can apply a type iii map to continue our sequence of divisions. At this stage we may introduce meromorphic coefficients by dividing by the leading coefficient of ${}_{l+2}f_{0,k}$. We may continue this until for some N we have $\min_i \operatorname{ord}{}_Nf_{i,k} = 1$ for all k. A can now be described as a union of two sets.

A_1: the zeros and poles of the leading coefficient of any ${}_lf_{i_0,k}$ with $\operatorname{ord}{}_lf_{i_0,k} = \min_i \operatorname{ord}{}_lf_{i,k}$.

A_2: the zero and poles of ${}_lf_{i,k_2} - {}_lf_{i,k_1}$ for all l, i, k_1, k_2 for which the function is not zero in K.

By removing A_1 we remove all the points where the maps ${}_lf_{i,k}$ have meromorphic coefficients. In fact, if we restrict attention to a neighborhood of $q \in \operatorname{Sg} V \cap U - A_1$, then for each of the images $V_{l,k}$ of the maps y, $(y_1,\cdots,y_{n-1},{}_lf_{n,k}(y),\cdots,{}_lf_{m-n+1,k}(y))$ are weakly equisingular analytic sets. Since we used the same operations on all $({}_lf_{i,k})$ which intersected formally at p, it is easy to see the maps $F_{l,k}$ whose inverses defined our operations give a resolution near any $q \in \operatorname{Sg} V \cap U - (A_1 \cup A_2)$. This completes the proof.

Now let us return to the case when V is residually equisingular at p. Let (x_i) be coordinates with respect to which a resolution is defined and $V_q = \{x \in U: x_i = q_i$ for $i < n\}$ for $q \in \operatorname{Sg} V \cap U$. Set $V_{k,q} = V_k \cap V_q$. From the proof of Proposition 1 it is easy to see that $(V_{k,q})$ are the global branches of V_q. By restriction a resolution of V will determine a resolution of each V_q. We denote by ${}_lV_q$ (resp. ${}_lV_{k,q}$) the inverse image of V_q (resp. $V_{k,q}$) under F_L. For each l $({}_lV_{k,q})$ continue to be the global branches of V_q.

Definition 4. For $q_1, q_2 \in \operatorname{Sg} V \cap U$ a bijective map $S:(V_{k,q_1}) \to (V_{k,q_2})$ is *tangentially stable* if

a) V_{k,q_1} and $S(V_{k,q_1})$ have the same multiplicity at t_1 and t_2 respectively.

b) For all k_1, k_2 we have $C_3(V_{k_1,q_1}, q_1) = C_3(V_{k_2,q_1}, q_1)$ if and only if $C_3(S(V_{k_1,q_1}), q_2) = C_3(S(V_{k_2,q_1}), q_2)$.

There is a natural map S to consider, $S(V_{k,q_1}) = V_{k,q_2}$. This map extends to (S_l) via $S_l({}_lV_{k,q_1}) = {}_lV_{k,q_2}$. It is easy to see from the proof of Proposition 1 that all the maps S_l satisfy condition 3a. The multiplicity can be computed directly from the normalizations constructed in that proof. Further the maps S_l respect the division of V_l into connected components, and restricting to a given component we get maps satisfying 4b. The proof of this last fact is contained in the following lemma, since at every stage ${}_lV$ is always weakly equisingular.

Lemma 5. *If V is weakly equisingular at p, then for $q_1, q_2 \in \operatorname{Sg} V$ near p we have $C_3(V_{k_1,q_1}, q_1) = C_3(V_{k_2,q_1}, q_1)$ if and only if $C_3(V_{k_1,q_2}, q_2) = C_3(V_{k_2,q_2}, q_2)$.*

Proof. By direct calculation, using the Puiseux series normalization for V, one sees that the two equalities mean, respectively, that the quadratic transforms of V_k and V_{k_2} meet at t_1 or t_2. Since the connected components of the transform of V are weakly equisingular we must have either both intersections or neither. This completes the proof.

Definition 6. Let $p \in \operatorname{Sg} V$ be such that $\operatorname{Sg} V$ is an $n-1$ manifold near p. We say that V is *strongly equisingular* along $\operatorname{Sg} V$ at p if $\dim C_5(V, p) = n+1$ and $\dim C_4(V, p) = n$.

From [4] we know that $\{x \in \operatorname{Sg} V \cap U : \dim C_5(V, x) > n+1\}$ is an analytic subset of codimension $\geqq 1$. As we mentioned in the introduction, no such description is known for $\{x \in \operatorname{Sg} V \cap U : V \text{ is not residually equisingular at } x\}$.

Example 1: A variety which is residually but not strongly equisingular at (0). Define $f: \mathbf{C}^2 \to \mathbf{C}^4$ by $f(y_1, y_2) = (y_1, y_2^6, y_2^8 + y_1, y_2^9, y_2^{13})$ and set $V = f(\mathbf{C}^2)$. f is proper so V is an analytic set. It is easy to check that $f: \mathbf{C}^2 \to V_1$ gives a Puiseux series normalization of V and so V is weakly equisingular at every point of $\operatorname{Sg} V = \{x \in \mathbf{C}^4 : x_i = 0 \text{ for } i \leqq 2\}$. (As in [3] we get $C_4(V_1, 0) = \{x \in \mathbf{C}^4 : x_i = 0 \text{ for } i > 2\}$ by direct computation.) Define $f_1: \mathbf{C}^2 \to \mathbf{C}^4$ by $f_1(y_1, y_2) = (y_1, y_2^6, y_2^2 + y_1 y_2^3, y_2^5)$ and set $V_1 = f_1(\mathbf{C}^2)$.

V is the image of V_1 under a type iii map. Restrict the domain of (f_1) to a neighborhood N disjoint from $\{y \in \mathbf{C}^2\colon y_1 y_2 = -1\}$ and replace V, V_1 by $f(N)$, $f_1(N)$. Projection on (x_1, x_3) represents V_1 as a branched cover of a domain D in $\mathbf{C}^2$, branching along $\{(x_1, x_3) \in D\colon x_3 = 0\}$. As in [3] a simple covering space argument shows that by changing coordinates in N we can bring f_1 into the form

$$f_1(z_1, z_2) = (z_1, z_2^6 + z_2^7 g(z), z_2^2, z_2^7 + z_2^8 h(z)).$$

Using this form of f_1 it is clear that V is residually equisingular at 0. Finally we want to show $C_5(V, 0) = \mathbf{C}^4$. Select α, β, $\gamma \in \mathbf{C}$ and set

$$y_1 = \gamma y_2^4,\ \bar{y}_1 = \beta y_1^{13/4},\ \bar{y}_2 = -y_2 + \alpha y_2^8.$$

For y_2 small, (y_1, y_2) and $(\bar{y}_1, \bar{y}_2)$ will be in N. Further, if we write the coordinate functions of $f(\bar{y}_1, \bar{y}_2) - f(y_1, y_2)$ as power series in y_2 they will have leading terms $(\beta \gamma^{13/4}, y_2^{13}, -6\alpha y_2^{13}, -2\gamma y_2^{13}, -2t^{13})$. Using this it is immediate from the definition of C_5 that

$$(\beta\gamma^{13/4} - 6\alpha, 2\alpha, -2\gamma, -2) = \lim_{y_2 \to 0}\ f(\bar{y},_1 \bar{y}_2) - f(y_1, y_2)/y_1^{13} \in C_5(V, 0)$$

for any (α, β, γ). Thus V is not strongly equisingular at 0.

Example 2. A variety which is strongly but not residually equisingular at (0). Define $f\colon \mathbf{C}^2 \to \mathbf{C}^4$ by $f(y_1, y_2) = (y_1, y_2^5, y_2^7, y_1 y_2^8)$ and set $V = f(\mathbf{C}^2)$. As above f is a Puiseux series normalization and V is weakly equisingular at every point of $\operatorname{Sg} V = \{x \in \mathbf{C}^4\colon x_1 = 0 \text{ for } i \geqq 2\}$. We wish to show that $C_5(V, 0) \cap \{x \in \mathbf{C}^4\colon x_4 = 0\} = (0)$ for this will imply $\dim C_5(V, 0) = 3$ and therefore that V is strongly equisingular at 0. Let τ be any fifth root of unity. From the results of section 3 of [2] it follows that it is sufficient to check that for $y_2 \neq 0$ we have

$$\lim_{y_1, y_2 \to 0}\ f(y_1, \tau y_2) - f(y_1, y_2)/y_2^5 \in \{x \in \mathbf{C}^4\colon x_4 = 0\}$$

and this is clearly the case.

Finally, we wish to show that V is not residually equisingular at 0. Suppose on the contrary, that a resolution of the required type exists, then as in the proof of Proposition 1, we could construct a map $g\colon D \to V$ which is the composite of a nonsingular map with maps of type i, ii, and iii. Here D is a polydisk in $\mathbf{C}^2$ centered at 0. Since f and g normalize V near 0 there is a biholomorphic map h of D into $\mathbf{C}^2$ such that $f \circ h = g$. Clearly h must be of the form $h(y) = (y_1, cy_2 + \text{higher order terms in } y_2)$, so the power series expansion of the third coefficient function begins with

$c_1 y_1 y_2^8$. However it is easy to see that no such function could occur in a map constructed as in the proof of Proposition 1. This is a simple power series calculation whose details we leave to the reader.

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